Category Theory - Important Results

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This is a record of the important results we cover during the lectures we will have in the summer 2018. We will try to go over two sets of lecture notes by Mariusz Wodzicki. Our goal is to introduce the concept of categories and build enough familiarity with them to be able to see other mathematical concepts we know in a more categorical point of view.

1 Review

In this section, we will review some concepts that will be helpful in the study of category theory.

1.1 Operations on sets

We give formal definitions of common set operators, giving a bit of a taste of the language we will use.

Definition 1 (Union of sets)**.** Let X be a set, we can define the **union** operator like so:

$$\bigcup = A \mapsto \{x \in X \mid \exists S \in A, x \in S\} : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$$

Definition 2 (Intersection of sets)**.** Let X be a set, we can define the **intersection** operator like so:

$$\bigcap = A \mapsto \{x \in X \mid \forall S \in A, x \in S\} : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$$

Definition 3 (Difference of sets)**.** Let X be a set, we an define the **difference** operator like so:

$$\setminus = (S,T) \mapsto \{x \in X \mid x \in S \wedge x \notin T\} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$$

Definition 4 (Cartesian product)**.** Let $(X_i)_{i\in I}$, where I is some index set, be a family of sets, the Cartesian product of these sets is

$$\prod_{i\in I} X_i = \{(x_i)_{i\in I} \mid \forall i \in I, x_i \in X_i\}.$$

We can also see each element as a function $f : I \to \cup_{i\in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$.

If a family of set is closed under the three first operations, we call it a ring of sets.

Definition 5 (Ring of sets)**.** A non-empty family of sets R is called a **ring** of sets if for any two elements r and r', we have $r \cup r', r \cap r', r \setminus r' \in R$.

1.2 Classes vs. Sets

Several times in our coverage of category theory, we will need to use the concept of a class. It is very similar to that of a set and has one simple difference. While a set can contain another set, classes cannot contain other classes. This difference is necessary because some collections of objects can simply not form a set. Famous examples include the class of ordinal numbers which, by the Burali-Forti paradox, cannot be a set and the class of all sets that do not contain themselves which, by the Russel paradox, cannot be a set.

2 Introduction to categories

2.1 Basic definitions

Definition 6 (Oriented graph)**.** An **oriented graph** G consists of a class of nodes G_0, a class of arrows G_1 along with two functions $s, t : G_1 \to G_0$, so that each arrow $f \in G_1$ has a source $s(f)$ and a target $t(f)$.

Remark 7. The nodes can also be called vertices or objects while arrows are also known as morphisms in the context of categories.

Definition 8 (Paths)**.** A **path** in an oriented graph G is a sequence of arrows $(f_1, \ldots, f_k)$ that are **composable** in the sense that $t(f_i) = s(f_{i-1})$ for $i = 2, \ldots, k$. We will denote G_k to be the class of paths of length k and we often refer to G_2 simply as the class of composable arrows.

Remark 9. Note that the notation indicating the direction of the path does not translate well to what we usually think of as a path in a graph. The reason is that the arrows are more linked to the composition of functions than paths in graphs.

Definition 10 (Category)**.** An oriented graph C along with a map $\circ : C_2 \to C_1$ is a **category** if for any $(f, g, h) \in C_3$, we have $f \circ (g \circ h) = (f \circ g) \circ h$, namely, composition is associative.

Definition 11 (Unital category)**.** A category C is called **unital** if it is equipped with a map $u : C_0 \to C_1$ (for $A \in C_0$, we denote $u(A) = \mathrm{id}_A$) such that for any arrow $f : A \to B$, we have $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$.

Definition 12 (Hom sets)**.** Let C be a category and $A, B \in C_0$, we denote

$$\mathrm{Hom}_C(A, B) = \{f \in C_1 \mid s(f) = A \wedge t(f) = B\}.$$

Definition 13 (Small and discrete)**.** A category C is called **small** if the class of objects and morphisms is not proper (it is a set). It is called **discrete** if there are no morphisms and **discrete unital** if there are no morphisms other than the identity morphisms.

Definition 14 (Subcategory)**.** Let C be a category, a category C' is a **subcategory** of C if:

1. The objects and morphisms of C' are objects and morphisms of C (i.e.: $C'_0 \subseteq C_0$ and $C'_1 \subseteq C_1$).
2. For every morphism $f \in C'_1$, $s(f), t(f) \in C'_0$.
3. For every pair of composable arrows $(f, g) \in C'_2$, $f \circ_{C'} g = f \circ_C g \in C'_1$.

If we are working with unital categories we have the additional requirement that for any $A \in C'_0$, $u_{C'}(A) \in C'_1$. One can show that since composition is the same as in C, the identity must be the same.

Definition 15 (Full and wide)**.** A subcategory C' of C is called **full** if for any objects $A, B \in C'_0$, we have $\mathrm{Hom}_{C'}(A, B) = \mathrm{Hom}_C(A, B)$. It is called **wide** if $C'_0 = C_0$.

Definition 16 (Covariant functor)**.** Let C and D be categories, a **covariant functor** $F : C \rightsquigarrow D$ is a pair of maps $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ that are defined such that the following diagrams commute (where F_2 is induced by the definition of F_1 with $(f, g) \mapsto (F_1(f), F_1(g))$).

$$\begin{tikzcd} C_0 \arrow[d, "F_0"'] & C_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"'] \\ D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r, "F_2"] \arrow[d, "\circ_C"'] & D_2 \arrow[d, "\circ_D"'] \\ C_1 \arrow[r, "F_1"'] & D_1 \end{tikzcd}$$

If we are working with unital categories, we may want to talk about a **unital** functor which requires this additional diagram to commute.

$$\begin{tikzcd} C_0 \arrow[r, "F_0"] \arrow[d, "u_C"'] & D_0 \arrow[d, "u_D"'] \\ C_1 \arrow[r, "F_1"'] & D_1 \end{tikzcd}$$

Definition 17 (Contravariant functor)**.** Let C and D be categories, a **contravariant functor** $F : C \rightsquigarrow D$ is similar to a covariant functor except for the first diagram which changes a bit (see below) and the definition of F_2 which becomes: $(f, g) \mapsto (F_1(g), F_1(f))$.

$$\begin{tikzcd} C_0 \arrow[d, "F_0"'] & C_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"'] \\ D_0 & D_1 \arrow[l, "t"] \arrow[r, "s"'] & D_0 \end{tikzcd}$$

Example 18 (Hom functors)**.** Let C be a category and $A \in C_0$ one of its object. We define the covariant and contravariant Hom functors from C to **Set**.

A. The functor $\mathrm{Hom}_C(A, -) : C \rightsquigarrow \mathbf{Set}$ sends an object $B \in C_0$ to the hom set $\mathrm{Hom}_C(A, B)$ and a morphism $f : B \to B'$ to the function

$$\mathrm{Hom}_C(A, f) : \mathrm{Hom}_C(A, B) \to \mathrm{Hom}_C(A, B') = g \mapsto f \circ g.$$

Let us check that this is a covariant functor. We show the commutativity of the three squares in definition 16:

1. For $f \in C_1$, $\mathrm{Hom}_C(A, s(f)) = s(\mathrm{Hom}_C(A, f))$ follows from the definition.
2. For $f \in C_1$, $\mathrm{Hom}_C(A, t(f)) = t(\mathrm{Hom}_C(A, f))$ follows from the definition.
3. For $(f_1, f_2) \in C_2$, we claim that $\mathrm{Hom}_C(A, f_1 \circ f_2) = \mathrm{Hom}_C(A, f_1) \circ \mathrm{Hom}_C(A, f_2)$. In the L.H.S., an element $g \in \mathrm{Hom}_C(A, s(f_1 \circ f_2))$ is mapped to $(f_1 \circ f_2) \circ g$ and in the R.H.S., an element $g \in \mathrm{Hom}_C(A, s(f_2)$ is mapped to $f_1 \circ (f_2 \circ g)$. Since $s(f_1 \circ f_2) = s(f_2)$, we see that the two maps are the same.

B. The functor $\mathrm{Hom}_C(-, A) : C \rightsquigarrow \mathbf{Set}$ sends an object $B \in C_0$ to the hom set $\mathrm{Hom}_C(B, A)$ and a morphism $f : B \to B'$ to the function

$$\mathrm{Hom}_C(f, A) : \mathrm{Hom}_C(B', A) \to \mathrm{Hom}_C(B, A) = g \mapsto g \circ f.$$

Let us check that this is a contravariant functor. We show the commutativity of the three squares in definition 17:

1. For $f \in C_1$, $\mathrm{Hom}_C(s(f), A) = s(\mathrm{Hom}_C(A, f))$ follows from the definition.
2. For $f \in C_1$, $\mathrm{Hom}_C(t(f), A) = t(\mathrm{Hom}_C(f, A))$ follows from the definition.
3. For $(f_1, f_2) \in C_2$, we claim that $\mathrm{Hom}_C(f_1 \circ f_2, A) = \mathrm{Hom}_C(f_2, A) \circ \mathrm{Hom}_C(f_1, A)$. In the L.H.S., an element $g \in \mathrm{Hom}_C(t(f_1 \circ f_2), A)$ is mapped to $g \circ (f_1 \circ f_2)$ and in the R.H.S., an element $g \in \mathrm{Hom}_C(t(f_1), A)$ is mapped to $(g \circ f_1) \circ f_2$. Since $t(f_1 \circ f_2) = t(f_1)$, we see that the two maps are the same.

Definition 19 (Full, faithfull and essentially surjective)**.** Let $F : C \rightsquigarrow D$ be a functor, then:

- If the restriction $F_{A,B} : \mathrm{Hom}_C(A, B) \to \mathrm{Hom}_D(F(A), F(B))$ is injective for any $A, B \in C_0$, then we say F is faithfull.
- If $F_{A,B}$ is surjective for any $A, B \in C_0$, then F is full.

- If for any $X \in D$, there exists $Y \in C_0$ such that $D \cong F(Y)$, then F is essentially surjective.

Definition 20 (Diagram)**.** Let C be a category. A **diagram** in C is functor $F : D \to C$ where D is usually a small or even finite category. We usually draw diagrams by partially drawing the image of D as a graph where objects are vertices and morphisms are arrows. All the diagrams we have drawn up to this definition define the domain of the functor implicitly. For example, if we talk about a commutative square in C, the domain of this diagram can be drawn like so:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Remark 21. It follows trivially from this definition that functors preserve commutative diagrams.

Definition 22 (Natural transformation)**.** Let $F, G : C \rightsquigarrow D$ be two covariant functors, a **natural transformation** $\phi : F \Rightarrow G$ is a map $\phi : C_0 \to D_1$ that satisfies $\phi(A) \in \mathrm{Hom}_D(F(A), G(A))$ for all $A \in C_0$ and makes the following diagram commute for any $f \in \mathrm{Hom}_C(A, B)$:

$$\begin{tikzcd} F(A) \arrow[r, "\phi(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\phi(B)"'] & G(B) \end{tikzcd}$$

For two contravariant functors, the vertical arrows are reversed.

Example 23. Let **CRing** denote the category of commutative rings, where objects are commutative rings, morphisms are ring homomorphisms, and composition is the usual composition of functions. Let **Grp** denote the category of groups, where objects are groups, morphisms are group homomorphisms, and composition is the usual composition of functions.

Fix some $n \in \mathbb{N}$, we define the functor $\mathrm{GL}_n : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$ by

$$R \mapsto \mathrm{GL}_n(R) \text{ for any commutative ring } R \text{ and}$$
$$f \mapsto \mathrm{GL}_n(f) \text{ for any ring homomorphism } f$$

The map $\mathrm{GL}_n(f)$ is just the extension of f on $\mathrm{GL}_n(R)$ by applying f to every element of the matrices. The second functor is $(-)^\times : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$ which sends a commutative ring R to its group of units $R^\times$ under multiplication and a ring homomorphism f to $f^\times$, its restriction on $R^\times$. Checking these mappings define two covariant functors is left as an (simple) exercise, but one might expect these to be functors as they play nicely with the structure of the objects involved.

The natural transformation between these two functors is $\det : \mathrm{GL}_n \Rightarrow (-)^\times$ which maps a commutative ring R to $\det_R$, the function calculating the determinant of a matrix in $\mathrm{GL}_n(R)$. The first thing to check is that $\det_R \in \mathrm{Hom}_{\mathbf{Grp}}(\mathrm{GL}_n(R), R^\times)$ which is clearly the case because the determinant of an invertible matrix is always a unit. The second thing is to verify that the following diagram commutes for any $f \in \mathrm{Hom}_{\mathbf{CRing}}(R, S)$:

$$\begin{tikzcd}
\mathrm{GL}_n(R) \arrow[r, "\det_R"] \arrow[d, "\mathrm{GL}_n(f)"'] & R^\times \arrow[d, "f^\times = f|_{R^\times}"] \\
\mathrm{GL}_n(S) \arrow[r, "\det_S"'] & S^\times
\end{tikzcd}$$

We will check the claim for $n = 2$, but the general proof should only involve more notation to write the bigger expressions. We can rewrite the diagram as $f^\times \circ \det_R = \det_S \circ \mathrm{GL}_2(f)$ and show it holds as follows. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(R)$, we have

$$\begin{aligned}
(\det{}_S \circ \mathrm{GL}_2(f)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) &= \det{}_S \left(\begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix} \right) \\
&= f(a)f(d) - f(b)f(c) \\
&= f(ad - bc) \\
&= f^\times(ad - bc) \\
&= (f^\times \circ \det{}_R) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).
\end{aligned}$$

We conclude that the diagram commutes and that det is indeed a natural transformation.

Definition 24 (Vertical composition)**.** Let $F, G, H : C \rightsquigarrow D$ be parallel functors and $\phi : F \Rightarrow G$ and $\psi : G \Rightarrow H$ be two natural transformations. Then the **vertical composition** of ϕ and ψ, denoted $\psi \cdot \phi : F \Rightarrow H$ is defined by $(\psi \cdot \phi)(A) = \psi(A) \circ \phi(A)$ for all $A \in C_0$. If $f : A \to B$ is a morphism in C, then we have the following diagram that commutes by naturality of ϕ and ψ:

$$\begin{tikzcd}
F(A) \arrow[r, "\phi(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\psi(A)"] \arrow[d, "G(f)"'] & H(A) \arrow[d, "H(f)"'] \\
F(B) \arrow[r, "\phi(B)"'] & G(B) \arrow[r, "\psi(B)"'] & H(B)
\end{tikzcd}$$

This shows that $\psi \cdot \phi$ is a natural transformation from F to H. We call this vertical composition as opposed to horizontal composition that we introduce in definition 66.

Definition 25 (Opposite category)**.** Let C be a category, we denote the **opposite category** C^{op} and define it by

$$C_0^{\mathrm{op}} = C_0, C_1^{\mathrm{op}} = C_1, s^{\mathrm{op}} = t, t^{\mathrm{op}} = s,$$

with the correspondence defined by $f^{\mathrm{op}} \circ^{\mathrm{op}} g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$. This canonically leads to the following contravariant functor $(-)^{\mathrm{op}}_C : C \rightsquigarrow C^{\mathrm{op}}$ which sends an object A to A^{op} and a morphism f to f^{op}. Note that the $^{\mathrm{op}}$ notation here is just used to distinguish elements in C and C^{op} although the class of objects and morphisms are the same.

Remark 26. The last definition helps us define the contravariant functors as covariant functors. Formally, let $F : C \rightsquigarrow D$ be a contravariant functor, we can see F as covariant functor from C^{op} to D or from C to D^{op} via the compositions $F \circ (-)^{\mathrm{op}}_{C^{\mathrm{op}}}$ and $(-)^{\mathrm{op}}_D \circ F$ respectively.

Definition 27 (Opposite of a functor)**.** Let $F : C \rightsquigarrow D$ be a covariant functor, then the **opposite** of this functor $F^{\mathrm{op}} : C^{\mathrm{op}} \rightsquigarrow D^{\mathrm{op}}$ is defined by $F^{\mathrm{op}} = (-)^{\mathrm{op}}_D \circ F \circ (-)^{\mathrm{op}}_{C^{\mathrm{op}}}$.

Definition 28 (Opposite functor)**.** The **opposite functor** $(-)^{\mathrm{op}} : \mathbf{Cat} \rightsquigarrow \mathbf{Cat}$ sends a category or a functor to its opposite. It is a covariant functor.

Definition 29 (Monomorphism)**.** Let C be a category, a morphism $f \in C_1$ is said to be a **monomorphism** if for any two morphisms $g, h \in C_1$ with $t(g) = t(h) = s(f)$, $f \circ g = f \circ h$ implies $g = h$.

Definition 30 (Epimorphism)**.** Let C be a category, a morphism $f \in C_1$ is said to be an **epimorphism** if for any two morphisms $g, h \in C_1$ with $s(g) = s(h) = t(g)$, $g \circ f = h \circ f$ implies $g = h$.

Proposition 31. *Let C be a category and $f : A \to B$ a morphism, if there exists $f' : B \to A$ such that $f' \circ f = id_A$, then f is a monomorphism.*

Proof. If $f \circ g = f \circ h$, then $f' \circ f \circ g = f' \circ f \circ h$ implying $g = h$. □

Proposition 32. *Let C be a category and $(f_1, f_2) \in C_2$, if $f_1 \circ f_2$ is a monomorphism, then f_2 is a monomorphism.*

Proof. Let $g, h \in C_1$ be such that $f_2 \circ g = f_2 \circ h$, we immediately get that $(f_1 \circ f_2) \circ g = (f_1 \circ f_2) \circ h$. Since $f_1 \circ f_2$ is a monomorphism, this implies $g = h$. □

Remark 33. The two dual propositions for epimorphisms also hold and are straightforward to prove.

Example 34 (Monomorphisms in the categories we know)**.**

1. Inside the category **Mon** where objects are monoids and morphims are monoid homomorphisms, the monomorphisms correspond exactly to injective homomorphims as shown below.

- Let $f : M \to M'$ be an injective homomorphims and $g_1, g_2 : N \to M$ be two parallel homomorphisms. Suppose that $f \circ g_1 = f \circ g_2$, then for all $x \in N$, $f(g_1(x)) = f(g_2(x))$, so by injectivity of f, $g_1(x) = g_2(x)$. We conclude that $g_1 = g_2$ and since g_1 and g_2 were arbitrary, f is a monomorphism.
- Let $f : M \to M'$ be a monomorphism. Let $x, y \in M$ and define $p_x : \mathbb{N} \to M$ by $k \mapsto x^k$ and similarly for p_y. It is trivial to show that p_x and p_y are homomorphism. If $f(x) = f(y)$, then by the homomorphism property, we get for all $k \in \mathbb{N}$:

  $$f(p_x(k)) = f(x^k) = f(x)^k = f(y)^k = f(y^k) = f(p_y(k)).$$

  In other words, we get $f \circ p_x = f \circ p_y$, so $p_x = p_y$ and $x = y$. We conclude that f is injective.

Example 35 (Epimorphisms in the categories we know)**.**

1. Inside the category **Mon** an epimorphism is not necessarily surjective. For example, the inclusion homomorphism $i : \mathbb{N} \to \mathbb{Z}$ is clearly not surjective but it is an epimorphism. Indeed, let $g, h : \mathbb{Z} \to M$ be two monoid homomorphisms satisfying $g \circ i = h \circ i$. In particular, we have $g(n) = h(n)$ for any $n \in \mathbb{N} \subset \mathbb{Z}$. It is left to show that also $g(-n) = h(-n)$, but if it were not the case for some n, $g(n)$ would have two left inverses $g(-n)$ and $h(-n)$ which is not possible. We conclude that $g = h$ and i is an epimorphism.

Definition 36 (Isomorphism)**.** Let C be a category, a morphism $f : A \to B$ is said to be an **isomorphism** if there exists a morphism $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

Proposition 37. *Let C be a category and $f \in C_1$ be an isomorphism, then f is a monomorphism and an epimorphism.*

Proof idea. If the compositions with f and two other morphisms are equal, compose with f^{-1} to obtain equality of the morphisms. □

Definition 38 (Natural isomorphism)**.** Let $\phi : F \to G$ be a natural transformation of functors $F, G : C \rightsquigarrow D$. If for every $A \in C_0$, $\phi(A)$ is an isomorphism in G, we say that ϕ is a **natural isomorphism** and we may write $\phi : F \cong G$.

Definition 39 (Equivalence of categories)**.**

Definition 40 (Subobject)**.**

Definition 41 (Quotient object)**.**

Definition 42 (Initial object)**.** Let C be a category, an object $A \in C_0$ is said to be **initial** if for any $B \in C_0$, $|\mathrm{Hom}_C(A, B)| = 1$, namely there are no two parallel morphisms with source A and every object has a morphism coming from A.

Definition 43 (Terminal object)**.** Let C be a category, an object $A \in C_0$ is said to be **terminal** if for any $B \in C_0$, $|\operatorname{Hom}_C(B, A)| = 1$, namely there are no two parallel morphisms with target A and every object has a morphism going to A.

Definition 44 (Zero object)**.** If an object is initial and terminal, we say it is a zero object and usually denote it 0.

Examples 45. We give examples of categories where initial and terminal objects may or may not exist.

1. $\exists$ terminal, $\nexists$ initial: Let **Sets'** denote the categories where objects are finite sets (excluding the empty set) and morphisms are surjective functions. Clearly, $\{1\}$ is final as any set can only map into $\{1\}$ by sending all their elements to 1. Suppose that a set S were initial, then it could be mapped surjectively to any other set T, implying that $|S| \geq |T|$ for any T. However, no finite number can be bigger than any other finite number, so we have a contradiction.

2. $\nexists$ terminal, $\exists$ initial: The category **GrpI** where the objects are groups and the morphisms are injective homomorphisms only contains an initial object $\{1\}$. Indeed, an injective homomorphism $G \to H$ can be seen as subgroup of H isomorphic to G. The identity group $\{1\}$ can only be isomorphic to the the identity subgroup as any other element has degree more than 1, so $\{1\}$ is initial. Moreover, a group G cannot be terminal as $G \times (\mathbb{Z}/2\mathbb{Z})$ cannot be isomorphic to any subgroup of G.

3. $\nexists$ terminal, $\nexists$ initial: Let G be a non trivial group. The category $G*$ has a single object $*$ with $\hom_{G*}(*, *) = G$ and the composition rule being the multiplication in G. The only object $*$ cannot be initial nor trivial as $|\hom_{G*}(*, *)| > 1$.

4. $\exists$ terminal, $\exists$ initial: Let X be a topological space where τ is the collection of open sets (recall that it must contain $\varnothing$ and X). We consider the category T_X where objects are the open sets and for any two open sets $U, V \in \tau$,

$$\hom_{T_X}(U, V) = \begin{cases} i_{U,V} & U \subseteq V \\ \varnothing & U \nsubseteq V \end{cases}$$

   Note that the composition rule can easily be inferred. Since the empty set is contained in every open set, it is an initial object. Since the full set X contains every open set, it is a terminal object. No other set can be initial as it cannot be contained in $\varnothing$ nor be terminal as it cannot contain X. Moreover, note that the two objects are not isomorphic as $\hom_{T_X}(X, \varnothing) = \varnothing$.

The following gives alternate definitions for initial and terminal objects which have the advantage of being completely categorical, making no use of sets. They use the concept of representable functors which will be seen more in depth later.

Proposition 46. *Let C be a category and $\star : C \to$ **Set** be a functor sending objects to the singleton $\{1\}$ and morphisms to $id_{\{1\}}$. An object $A \in C_0$ is initial if and only if the functor $\operatorname{Hom}_C(A, -)$ is naturally isomorphic to $\star$.*

Proof. ($\Rightarrow$) Suppose that A is initial, then there is a natural transformation η from $\hom_C(A, -)$ to $\star$ that sends any object X to the only function between $\hom_C(A, X)$ and $\{1\}$. Since the $\hom_C(A, X)$ is also a singleton, this function is an isomorphism for all X and we conclude that η is a natural isomorphism.

($\Leftarrow$) Suppose that there is a natural isomorphism $\eta : \hom_C(A, -) \Rightarrow \star$, then there are isomorphisms between $\{1\}$ and $\hom_C(A, X)$ for all objects $X \in C_0$. This means that there is a unique morphism from A to X and that A is initial. $\square$

Proposition 47. *Let C be a category and $\star$ be as above. An object $A \in C_0$ is terminal if and only if the functor $\mathrm{Hom}_C(-, A)$ is naturally isomorphic to $\star$.*

Proof. The proof is basically a copy of the last proof. $\square$

Proposition 48. *Let C be a category, A and B are two initial (this also works for terminal) objects of C, then $A \cong B$.*

Proof. Let f be the single element in $\hom_C(A, B)$ and f' be the single element in $\hom_C(B, A)$. We claim that f and f' are inverses, thus that $A \cong B$. Since the identity morphisms are the only elements of $\hom_C(A, A)$ and $\hom_C(B, B)$, and $f' \circ f$ and $f \circ f'$, respectively, are elements of these sets, they must be the identities. $\square$

Definition 49 (Product)**.** Let C be a category and $A, B \in C_0$. A **product** of A and B is an object denoted $A \times B$ along with two morphisms $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ (they are called projections) such that for any object V and morphisms $f : V \to A$ and $g : V \to B$, there exists a unique morphism $h : V \to A \times B$ such that this diagram commutes:

$$\begin{tikzcd}
 & V \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

Example 50. Inside **Set**, the Cartesian products with the usual projection maps are products. Inside **Grp**, the direct products with the usual projection maps are products.

Definition 51 (Coproducts)**.** Let C be a category and $A, B \in C_0$. A **coproduct** of A and B is an object denoted $A \amalg B$ along with two morphisms $i_1 : A \to A \times B$ and $i_2 : B \to A \times B$ (they are called canonical injections) such that for any object V and morphisms $f : A \to V$ and $g : B \to V$, there exists a unique morphism $h : A \times B \to V$ such that this diagram commutes:

$$\begin{tikzcd}
 & V & \\
A \arrow[ur, "f"] \arrow[r, "i_1"'] & A \times B \arrow[u, dotted, "h"] & B \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Definition 52 (Pullback)**.** Let C be a category and $f : A \to C$ and $g : B \to C$ be in C_1. A **pullback** of f and g is an object denoted $A \times_C B$ along with two morphisms $p_1 : A \times_C B \to A$ and $p_2 : A \times_C B \to B$ such that this diagram commutes: and for any

$$\begin{tikzcd} A \times_C B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

object V and morphisms $s : V \to A$ and $t : V \to B$, there exists a unique morphism $h : V \to A \times_C B$ that makes this diagram commute:

$$\begin{tikzcd} V \arrow[dr, dotted, "h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & A \times_C B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

Definition 53 (Pushout)**.**

Example 54 (Pushouts in **Grp**)**.** Let $f : A \to B$ and $g : A \to C$ be group homomorphism. We construct the pushout X. We let X be the group generated by all elements in $B \amalg C$ subject to the following relations for all generators $x, y, z \in X$:

- If x, y in the same group and $z = (xy)^{-1}$, then $xyz = 1$.
- If $x = f(a)$ and $y = g(a)^{-1}$, then $xy = 1$ or $x = g(a)$ and $y = f(a)^{-1}$.

We already have the other arrows of the square being the inclusion maps $B \to X$ and $C \to X$. We just need to check commutativity but the second relation helps with that. For any other M in a square, define $q : X \to M$ by sending a generator to the image of the generator under the arrows of the square with M. Look at what $\mathbb{Z}_2 * \mathbb{Z}_2$ is $\{x, y, xy, yx, xyx, ...\}$.

Question 55. *Is the pullback object always a subobject of the product ? Is the pushout object always a subobject of the coproduct or quotient object of the product ? Why are these terms used ?*

Definition 56 (Equalizer)**.** Let C be a category, $X, Y \in C_0$ and $f, g \in \mathrm{Hom}_C(X, Y)$ be distinct. The equalizer of f and g is an object E and a morphism $e : E \to X$ such that $f \circ e = g \circ e$ and this universal property is satisfied: if $o : O \to X$ is such that $f \circ o = g \circ o$, then there exists a unique morphism $u : O \to E$ such that $e \circ u = o$. In picture, we have

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \\
O \arrow[u, dashed, "u"] \arrow[ur, "o"'] & &
\end{tikzcd}$$

Definition 57 (Co-equalizer)**.** Let C be a category, $X, Y \in C_0$ and $f, g \in \mathrm{Hom}_C(X, Y)$ be distinct. The co-equalizer of f and g is an object D and a morphism $d : Y \to D$ such that $d \circ f = d \circ g$ and this universal property is satisfied: if $o : X \to O$ is such that $o \circ f = o \circ g$, then there exists a unique morphism $u : D \to O$ such that $u \circ d = o$. In picture, we have

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "d"] \arrow[dr, "o"'] & D \arrow[d, dashed, "u"] \\
 & & O
\end{tikzcd}$$

Definition 58 (Equivalence)**.** A functor $F : C \rightsquigarrow D$ is an equivalence of categories if there exists a functor $G : D \rightsquigarrow C$ such that $FG \cong \mathrm{id}_C$ and $GF \cong \mathrm{id}_D$, where $\cong$ denote natural isomorphism.

Theorem 59. *A functor $F : C \rightsquigarrow D$ is an equivalence of categories if and only if F is fully faithfull and essaentially surjective.*

2.2 More on natural transformations

Definition 60 (The left action of functors)**.** Let $F, F' : C \rightsquigarrow D$, $G : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. The functor G acts on ϕ by sending it to $G\phi = A \mapsto G(\phi(A)) : C_0 \to E_1$. One can verify that this is a natural transformation from $G \circ F$ to $G \circ F'$ by verifying the diagram commutes for any $C_1 \ni f : A \to B$.

$$\begin{array}{ccc}
(G \circ F)(A) & \xrightarrow{G\phi(A)} & (G \circ F')(A) \\
{\scriptstyle (G\circ F)(f)}\big\downarrow & & \big\downarrow{\scriptstyle (G\circ F')(f)} \\
(G \circ F)(B) & \xrightarrow{G\phi(B)} & (G \circ F')(B)
\end{array}$$

If we remove all applications of G, the diagram commutes by naturality of ϕ. Since functors preserve commuting diagrams, we get that $G\phi$ is a natural transformation.

Proposition 61. *The previous definition constitutes a left action, namely, $\mathrm{id}_D\phi = \phi$ and $G_1(G_2\phi) = (G_1 \cdot G_2)\phi$.*

Proof. $\square$

Definition 62 (The right action of functors)**.** Let $F, F' : C \rightsquigarrow D$, $G : E \rightsquigarrow C$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. The functor G acts on ϕ by sending it to $\phi G = A \mapsto \phi(G(A)) : E_0 \to D_1$. One can verify that this is a natural transformation from $F \circ G$ to $F' \circ G$ by verifying the diagram commutes for any $E_1 \ni f : A \to B$.

$$\begin{tikzcd}
(F \circ G)(A) \arrow[r, "\phi G(A)"] \arrow[d, "(F \circ G)(f)"'] & (F' \circ G)(A) \arrow[d, "(F' \circ G)(f)"] \\
(F \circ G)(B) \arrow[r, "\phi G(B)"] & (F' \circ G)(B)
\end{tikzcd}$$

It follows by naturality of ϕ; change f in the diagram of definition 22 with the morphism $G(f) : G(A) \to G(B)$.

Proposition 63. *The previous definition constitutes a right action, namely, $\phi id_C = \phi$ and $(\phi G_1)G_2 = \phi(G_1 \cdot G_2)$.*

Proof. □

Proposition 64. *The two actions commute. Namely, if we let $F, F' : C \rightsquigarrow D$, $G : D \rightsquigarrow E$, $H : E' \rightsquigarrow C$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation, then we have $G(\phi E) = (G\phi)E$.*

Proof. □

We will refer to these two actions as the biaction of functors on natural transformations and they will motivate the definition of another way to compose natural transformations. Consider the following diagram to be the setting of this definition, with F, F', G and

$$\begin{tikzcd}
C \arrow[r, bend left=60, "F", ""{name=U, below}] \arrow[r, bend right=60, "F'"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "\phi"] & D \arrow[r, bend left=60, "G", ""{name=U2, below}] \arrow[r, bend right=60, "G'"', ""{name=D2, above}] \arrow[Rightarrow, from=U2, to=D2, "\psi"] & E
\end{tikzcd}$$

G' being functors and ϕ and ψ being natural transformations. With the two previous actions, we are able to construct four new transformations:

$$G\phi : G \circ F \Rightarrow G \circ F'$$
$$\psi F : G \circ F \Rightarrow G' \circ F$$
$$G'\phi : G' \circ F \Rightarrow G' \circ F'$$
$$\psi F' : G \circ F' \Rightarrow G' \circ F'$$

Observe that to go from $G \circ F$ to $G' \circ F'$, we have two paths yielding the following diagram:

$$\begin{tikzcd}
 & G \circ F' \arrow[dr, "\psi F'"] & \\
G \circ F \arrow[ur, "G\phi"] \arrow[dr, "\psi F"'] & & G' \circ F' \\
 & G' \circ F \arrow[ur, "G'\phi"'] &
\end{tikzcd}$$

Proposition 65. *The diagram above commutes.*

Proof. For a fixed element $c \in C_0$ we know that $a := F(c)$ and $b := F'(c)$ are two different elements of the category D and that we have an arrow $f := \phi(c)$ from a to b given by the natural transformation ϕ. But as ψ is a natural transformation $G \Rightarrow G'$, we know that the following diagram commutes:

$$\begin{tikzcd}
 & G(b) \arrow[dr, "\psi b"] & \\
G(a) \arrow[ur, "Gf"] \arrow[dr, "\psi a"'] & & G'(b) \\
 & G'(a) \arrow[ur, "G'f"'] &
\end{tikzcd}$$

Replacing a, b and f by their values we obtain what we wanted. □

Definition 66 (Horizontal composition)**.** In the setting described above, we define the **horizontal composition** of ψ and ϕ by $\psi \diamond \phi = \psi F' \cdot G\phi = G'\phi \cdot \psi F$.

Proposition 67. *Horizontal composition is associative. Namely, if we let $F, F' : C_1 \rightsquigarrow C_2$, $G, G' : C_2 \rightsquigarrow C_3$ and $H, H' : C_3 \rightsquigarrow C_4$ be functors and $\phi : F \Rightarrow F'$, $\psi : G \Rightarrow G'$ and $\eta : H \Rightarrow H'$ be natural transformations, then we have $\eta \diamond (\psi \diamond \phi) = (\eta \diamond \psi) \diamond \phi$.*

Proof. □

Proposition 68 (Interchange identity)**.** *Let $F, F', F'' : C \rightsquigarrow D$ and $G, G', G'' : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$, $\phi' : F' \Rightarrow F''$, $\psi : G \Rightarrow G'$ and $\psi' : G' \Rightarrow G''$ be natural transformations. Using $\cdot$ to denote vertical composition, the* ***interchange identity*** *holds:*

$$(\psi' \cdot \psi) \diamond (\phi' \cdot \phi) = (\psi' \diamond \phi') \cdot (\psi \diamond \phi)$$

Proof. The idea is to use the commutativity of $\psi' \circ \phi$ to switch from the LHS to the RHS of the equation. To make things clearer we first draw out the diagrams The LHS of the equation can be seen as the following diagram:

While the RHS would correspond to the following:

Joining the two diagrams, we obtain this huge one □

$$
\begin{tikzcd}
 & G' \circ F \arrow[dr, "G'\phi"] & & G'' \circ F' \arrow[dr, "G''\phi'"] & \\
G \circ F \arrow[ur, "\psi F"] \arrow[dr] & & G' \circ F' \arrow[ur, "\psi' F'"] \arrow[dr] & & G'' \circ F'' \\
 & G \circ F' \arrow[ur] & & G' \circ F'' \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & G'' \circ F \arrow[dr, "G''\phi"] & & \\
 & G' \circ F \arrow[ur, "\psi' F"] & & G'' \circ F' \arrow[dr, "G''\phi'"] & \\
G \circ F \arrow[ur, "\psi F"] \arrow[dr] & & & & G'' \circ F'' \\
 & G \circ F' \arrow[dr] & & G' \circ F'' \arrow[ur] & \\
 & & G \circ F'' \arrow[ur] & &
\end{tikzcd}
$$

These definitions lead us to the first example of a 2-category.

Definition 69 (2-cateory)**.** A **2-category** consists of a class of objects C_0, a class of morphisms between objects C_1 and a class of 2-morphisms between parallel morphisms C_2 that satisfy the following conditions:

1. The objects and morphisms form a category under composition of morphisms.
2. For two objects $A, B \in C_0$, the morphisms from C to D and the 2-morphisms between them form a category under vertical composition.
3. If we consider 2-cells (two parallel morphisms with a 2-morphism between them) as morphisms, we get a category under horizontal composition.
4. The interchange identity hold for horizontal and vertical composition.

Example 70.

1. The 2-category of categories with functors and natural transformations as we just have proved.

Question 71. *Is the vertical composition of two natural isomorphisms also a natural isomorphism ? What about horizontal composition ?*

Definition 72 (Identity transformation)**.** Let $F : C \rightsquigarrow D$ be a functor, the identity natural transformation from F to itself is defined by $\mathrm{id}_F = A \mapsto \mathrm{id}_{F(A)} : C_0 \to D_1$ when the objects in the range of F all have an identity morphism.

$$
\begin{tikzcd}[column sep=small]
 & & G'' \circ F \arrow[dr, "G''\phi"] & & \\
 & G' \circ F \arrow[ur, "\psi' F"] \arrow[dr, "G'\phi"] & 1 & G'' \circ F' \arrow[dr, "G''\phi'"] & \\
G \circ F \arrow[ur, "\psi F"] \arrow[dr] & & G' \circ F' \arrow[ur, "\psi' F'"] \arrow[dr] & & G'' \circ F'' \\
 & G \circ F' \arrow[ur] \arrow[dr] & & G' \circ F'' \arrow[ur] & \\
 & & G \circ F'' \arrow[ur] & &
\end{tikzcd}
$$

Proposition 73. *Let $F, F' : C \rightsquigarrow D$, $G : B \rightsquigarrow C$ and $H : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. Suppose that C and E are unital, then the following equations hold:*

1. $\phi G = \phi \diamond id_G$
2. $H\phi = id_H \diamond \phi$
3. $id_{id_D} \diamond \phi = \phi = \phi \diamond id_{id_C}$

Proof.

1. For any $x \in B_0$, we have the following:

$$
\begin{aligned}
(\phi \diamond \mathrm{id}_G)(x) &= \phi(G(x)) \circ F(\mathrm{id}_G(x)) && \text{(def of } \diamond\text{)} \\
&= \phi G(x) \circ F(\mathrm{id}_{G(x)}) && \text{(def of id}_G\text{)} \\
&= \phi G(x) \circ \mathrm{id}_{F(G(x))} && \text{(functors preserve id morphisms)} \\
&= \phi G(x) && \text{(def of id morphisms)}
\end{aligned}
$$

Thus, we conclude that $\phi \diamond \mathrm{id}_G = \phi G$.

2. For any $x \in C_0$, we have the following:

$$
\begin{aligned}
(\mathrm{id}_H \diamond \phi)(x) &= \mathrm{id}_H(F'(x)) \circ H(\phi(x)) && \text{(def of } \diamond\text{)} \\
&= \mathrm{id}_{H(F'(x))} \circ H\phi(x) && \text{(def of id}_H\text{)} \\
&= H\phi(x) && \text{(def of id morphisms)}
\end{aligned}
$$

Thus, we conclude that $\mathrm{id}_H \diamond \phi = H\phi$.

3. By swapping G for id_C and H for id_D in the two previous equations, we get the result we want.

□

2.3 On our way to the Yoneda lemma

Definition 74 (Category of arrows)**.** Let C be a category, $\mathrm{Arr}(C)$ is the category of arrows of C. Its objects are morphisms in C and its morphisms are commutative squares ϕ. In other words, if f and g are morphisms in C and there exists maps ϕ_s and ϕ_t such that this diagram commutes

$$\begin{tikzcd} s(f) \arrow[r, "f"] \arrow[d, "\phi_s"'] & t(f) \arrow[d, "\phi_t"] \\ s(g) \arrow[r, "g"'] & t(g) \end{tikzcd},$$

then this square is a morphism from f to g. It is denoted by ϕ or (ϕ_s, ϕ_t).

Definition 75 (Source functor)**.** Let C be a category, the **source functor** is $S : \mathrm{Arr}\, C \rightsquigarrow C$ defined by:

$$\begin{aligned} S_0(f) &= s(f), \forall f \in C_1 = \mathrm{Arr}(C)_0 \\ S_1((\phi_s, \phi_t)) &= \phi_s \forall (\phi_s, \phi_t) \in \mathrm{Arr}(C)_1 \end{aligned}$$

Definition 76 (Target functor)**.** Let C be a category, the **target functor** is $T : \mathrm{Arr}\, C \rightsquigarrow C$ defined by:

$$\begin{aligned} T_0(f) &= t(f), \forall f \in C_1 = \mathrm{Arr}(C)_0 \\ T_1((\phi_s, \phi_t)) &= \phi_t \forall (\phi_s, \phi_t) \in \mathrm{Arr}(C)_1 \end{aligned}$$

Definition 77 (Tautological natural transformation)**.** Let C be a category, the **tautological natural transformation** is $\tau : S \Rightarrow T$ defined by $\tau(f) = f$ for all $f \in C_1 = \mathrm{Arr}(C)_0$. Note that we see the input as an object of $\mathrm{Arr}(C)$ and the output as a morphism of C.

Definition 78 (Arr functor)**.** The **Arr functor** is a functor **Cat** $\rightsquigarrow$ **Cat** that sends a category C to its category of arrows and a functor $F : C \rightsquigarrow D$ to the functor $\mathrm{Arr}(F) : \mathrm{Arr}(C) \rightsquigarrow \mathrm{Arr}(D)$ defined by

$$\begin{aligned} &\mathrm{Arr}(F)_0 = f \mapsto F(f) \\ &\mathrm{Arr}(F)_1 = (\phi_s, \phi_t) \mapsto (F(\phi_s), F(\phi_t)) \end{aligned}$$

Proposition 79. *The correspondences $S = C \mapsto S_C$ and $T = C \mapsto T_C$ where S_C is the source functor and T_C is the target functor define natural transformations* $\mathrm{Arr} \mapsto id_{\mathbf{Cat}}$.

Proof. □

Definition 80 (Representable functors)**.** A covariant functor $F : C \to$ **Set** is said to be representable if there is an object $X \in C_0$ such that F is naturally isomorphic to $\hom_C(X, -)$. If F is contravariant, then we require it to be naturally isomorphic to $\hom_C(-, X)$.

Example 81. The functor $(-)^{\times} : \textbf{Ring} \rightsquigarrow \textbf{Set}$ is represented by $\mathbb{Z}[x, x^{-1}]$ because any unit of $R^{\times}$ corresponds to the unique homomorphism from $\mathbb{Z}[x, x^{-1}]$ to R sending x to that unit and every homomorphisms from $\mathbb{Z}[x, x^{-1}]$ to R must send x to a unit.

Example 82. The forgetful functor has a left adjoint implies it is representable. Look at what happens on a set with one element. Need to define forgetful functor and adjoint.

Example 83 (Cayley's theorem with the Yoneda Lemma)**.** Cayley's theorem states that any group is isomorphic to the subgroup of a permutation group. We will use the Yoneda lemma to show that.

Recall the first part of the Yoneda lemma which states that for a category C, a functor $F : C \rightsquigarrow \textbf{Sets}$ and an object A. We have

$$\mathrm{Nat}(\mathrm{Hom}(A, -), F) \cong F(A).$$

Moreover, we know the explicit maps, namely, a natural transformation φ in the L.H.S. is mapped to $\varphi_A(\mathrm{id}_A)$ and an element $u \in F(A)$ is mapped to the natural transformation $\{\varphi_B = f \mapsto F(f)(u) \mid B \in C_0\}$.

Let us apply this to C being the category associated to a group G (i.e.: there is one object $\star$, $\mathrm{Hom}(\star, \star) = G$ and the composition law follows the group operation). Note that any functor $F : C \rightsquigarrow \textbf{Sets}$ sends $\star$ to a set S and any $g \in G$ to a permutation of S, otherwise $g \circ g^{-1} = 1$ cannot be satisfied.

To use the Yoneda lemma, our only choice for A is $\star$ and we will choose $F = \mathrm{Hom}(\star, -)$. The Yoneda correspondence becomes

$$\mathrm{Nat}(\mathrm{Hom}(\star, -), \mathrm{Hom}(\star, -)) \cong \mathrm{Hom}(\star, \star).$$

We already know what the R.H.S. is G, but we have to do a bit of work to understand the L.H.S. First, observe that a natural transformation $\varphi : \mathrm{Hom}(\star, -) \Rightarrow \mathrm{Hom}(\star, -)$ is just one morphism $\varphi_\star : \mathrm{Hom}(\star, \star) \to \mathrm{Hom}(\star, \star)$. Namely, it is a map from G to G. Second, recalling that $\mathrm{Hom}(\star, g) = g \circ (-)$ and that $\star$ is the only object in C_0, we get that $\varphi_\star$ must only satisfy one diagram.

$$\begin{tikzcd}
G \arrow[r, "\varphi_\star"] \arrow[d, "g\circ(-)"'] & G \arrow[d, "g\circ(-)"] \\
G \arrow[r, "\varphi_\star"'] & G
\end{tikzcd}$$

This is equivalent to $\varphi_\star(g \cdot h) = g \cdot \varphi_\star(h)$, and we get that each $\varphi_\star$ is a G-equivariant map, denote these maps $\mathrm{Hom}_G(G, G)$. We obtain

$$\mathrm{Hom}_G(G, G) \cong G.$$

Now, it is easy to check that $\mathrm{Hom}_G(G, G)$ is a subgroup of Σ_G (the group of permutations of the set G) and that the correspondence is in fact an isomorphism of groups. Cayley's theorem follows.

Let us check that $\mathrm{Hom}_G(G, G) < \Sigma_G$. Let f be a G-equivariant map. For any $g \in G$, we have $f(g) = f(g \cdot 1) = g \cdot f(1)$. Thus, f is determined only by where it sends the identity. Additionally, since $g \cdot f(1)$ ranges over G when g ranges over G, for any choice of $f(1)$, f is bijective. Finally, if f and f' are both G-equivariant map, then

$$(f \circ f')(g \cdot h) = f(f'(g \cdot h)) = f(g \cdot f'(h)) = g \cdot (f \circ f')(h),$$

hence $f \circ g$ is G-equivariant. With the facts that f^{-1} is just the G-equivariant map sending 1 to $f(1)^{-1}$ and id is G-equivariant, it follows that $\mathrm{Hom}_G(G, G)$ is a subgroup of Σ_G.

The final check is that the Yoneda correspondence $G \to \mathrm{Hom}_G(G, G)$ sending g to $(-) \cdot g$ is a group homomorphism (isomorphism follows because it is a bijection). It is clear that it sends the identity to the identity and for any $g, h \in G$

$$(-) \cdot gh = ((-) \cdot g) \cdot h = ((-) \cdot h) \circ ((-) \cdot g),$$

so this is a group homomorphism.

3 Limits

Definition 84 (Cones)**.** Let $F : J \rightsquigarrow C$ be a diagram in C and $X \in C_0$. A cone from X to F is a family $\{\psi_Y : X \to F(Y)\}_{Y \in J_0}$ such that for any morphism $f : Y \to Z$ in J, $F(f) \circ \psi_Y = \psi_X$, i.e.: the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dl, "\psi_Y"'] \arrow[dr, "\psi_Z"] & \\ F(Y) \arrow[rr, "F(f)"] & & F(Z) \end{tikzcd}$$

Definition 85 (Morphism of cones)**.** Let $F : J \rightsquigarrow C$ be a diagram in C and $\{\psi_Y : A \to F(Y)\}_{Y \in J_0}$ and $\{\phi_Y : B \to F(Y)\}_{Y \in J_0}$ be two cones to F. A morphism of cones from A to B is a morphism $g : A \to B$ in C such that for any Y, we have $\psi_Y \circ g = \phi_Y$, i.e.: the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "g"'] \arrow[dr, "\psi_Y"'] & & B \arrow[dl, "\phi_Y"] \\ & F(Y) & \end{tikzcd}$$

Definition 86 (Limit)**.** Let $F : J \rightsquigarrow C$ be a diagram in C, the limit of F (or J) denoted $\varprojlim F$ (or $\varprojlim J$), if it exists, is the terminal object of the category of cones to F.

Remark 87. Observe that products arises as limits of diagrams where the domain is discrete unital, i.e.: it has no morphisms but the identities.

Proposition 88. *Suppose that a category C has arbitrary products and equalizers then C has arbitrary limits.*

Proof. Let $F : J \rightsquigarrow C$ be a diagram, we claim that the the limit of F is the equalizer of

$$u_1, u_2 : \prod_{X \in J_0} F(X) \to \prod_{a \in J_1} F(t(a)),$$

where u_1 and u_2 are defined below. This equalizer and the products it involves exists by our hypothesis.

For any $X \in J_0$ and $a \in J_1$, we have the following projections

$$\pi_X : \prod_{X \in J_0} F(X) \to F(X) \qquad \pi_a : \prod_{a \in J_1} F(t(a)) \to F(t(a)).$$

Moreover, note that $\prod_{X \in J_0} F(X)$ has two different ways to project to $F(t(a))$ for all $a \in J_1$. The first one being via $\pi_{t(a)}$, we get a unique morphism $u_1 : \prod_{X \in J_0} F(X) \to \prod_{a \in J_1} F(t(a))$ that satisfies $\pi_a \circ u_1 = \pi_{t(a)}$. For the second one, note that $F(a) \circ \pi_{s(a)}$ is also a projection to $F(t(a))$, thus we get a unique morphism $u_2 : \prod_{X \in J_0} F(X) \to \prod_{a \in J_1} F(t(a))$ that satisfies $\pi_a \circ u_2 = F(a) \circ \pi_{s(a)}$.

Let $e : E \to \prod_{X \in J_0} F(X)$ be the equalizer of u_1 and u_2 and for any $X \in J_0$, let $\psi_X = \pi_X \circ e$. For any $f : Y \to Z$ in J, we have

$$\begin{aligned} F(f) \circ \psi_Y &= F(f) \circ \pi_Y \circ e \\ &= \pi_f \circ u_2 \circ e \\ &= \pi_f \circ u_1 \circ e \\ &= \pi_Z \circ e = \psi_Z, \end{aligned}$$

so we indeed obtain a cone from E to F. Next, for any other cone $\{\phi_X : O \to F(X)\}_{X \in J_0}$, we get a unique morphism $p : O \to \prod_{X \in J_0}$ such that $\pi_X \circ p = \phi_X$ (by universality of the product). We claim that both $u_1 \circ p$ and $u_2 \circ p$ make the following diagram commute for all $a \in J_1$.

$$\begin{tikzcd} O \arrow[r, "u_i \circ p"] \arrow[dr, "\phi_{t(a)}"'] & \prod_{a \in J_1} F(t(a)) \arrow[d, "\pi_a"] \\ & F(t(a)) \end{tikzcd}$$

We have the following derivations.

$$\pi_a \circ u_1 \circ p = \pi_{t(a)} \circ p = \phi_{t(a)}$$

$$\begin{aligned} \pi_a \circ u_2 \circ p &= F(a) \circ \pi_{s(a)} \circ p \\ &= F(a) \circ \phi_{s(a)} \\ &= \phi_{t(a)} \end{aligned}$$

By universality of the product of the $F(t(a))$'s, we obtain $u_1 \circ p = u_2 \circ p$ and by universality of the equalizer, we get a unique morphism $n : O \to E$ such that $e \circ n = p$. Furthermore, for any $X \in J_0$, we have

$$\psi_X \circ n = \pi_X \circ e \circ n = \pi_X \circ p = \phi_X,$$

so n is also a morphism of cones $(O, \phi_X) \to (E, \psi_X)$. Since any other morphism of cones m needs to satisfy $e \circ m = p$, we see that n is unique. We conclude that E is indeed the limit of F. $\square$

Remark 89. The same proof yields a more general statement: For any cardinal κ, if a category C has products of size κ and equalizers, then it has limits of any diagram with at most κ objects and morphisms.

4 Monads

Definition 90 (σ-algebra)**.**

Definition 91 (Measurable spaces)**.** A measurable space is a set X along with a σ-algebra of X. A function between two measurable spaces (X, Σ_X) and (Y, Σ_Y) (which is just a function $f : X \to Y$ is said to be measurable if the preimage of any measurable set is measurable. The category **Mes** has measurable spaces as its objects and measurable functions as its morphisms.

Definition 92 (Giry Monad)**.** We define the monad $\mathcal{G} : \mathbf{Mes} \rightsquigarrow \mathbf{Mes}$. It sends (X, Σ_X) to the set of probability measures on Σ with the smallest σ-algebra that makes e_A measurable for all $A \in \Sigma_X$, where

$$e_A : \mathcal{G}(X) \to [0,1] = p \mapsto p(A),$$

and $[0,1]$ has the usual Borel σ-algebra. For a morphism $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$, $\mathcal{G}(f)$ sends a measure to its image measure (or push-forward), namely,

$$\mathcal{G}(f) : \mathcal{G}(X) \to \mathcal{G}(Y) = p \mapsto p \circ f^{-1}.$$

It remains to define the two natural transformations $\mu : \mathcal{G}^2 \Rightarrow \mathcal{G}$ and $\eta : \mathrm{id}_{\mathbf{Mes}} \Rightarrow \mathcal{G}$. For the former, if (X, Σ_X) is measurable and $\Omega \in \mathcal{G}^2(X)$, we define

$$\mu_X(\Omega) : \Sigma_X \to [0,1] = A \mapsto \int_{p \in \mathcal{G}(X)} e_A(p) d\Omega.$$

For the latter, we define

$$\eta_X(x) : \Sigma_X \to [0,1] = \delta_x := A \mapsto \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Definition 93. Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces, a **Markov kernel** is a map $f : X \times \Sigma_Y \to [0,1]$ such that for any $B \in \Sigma_Y$, $f(\cdot, B) : X \to [0,1]$ is Σ_X-measurable ($[0,1]$ has the usual Borel σ-algebra) and for any $x \in X$, $f(x, \cdot)$ is a probability measure on Σ_Y. We define the compositions of two Markov kernels $f : X \times \Sigma_Y \to [0,1]$ and $g : Y \times \Sigma_Z \to [0,1]$ as

$$g \circ f : X \times \Sigma_Z \to [0,1] = (x, C) \mapsto \int_{y \in Y} g(y, C) f(x, dy).$$

In words, (x, C) is mapped to the average of $g(y, C)$ weighted by the measure $f(x, \cdot)$.

Proposition 94. *The category of Markov kernels is the Kleisli category of the Giry monad.*

Proof. Recall that in the Giry monad, morphisms are measurable functions $f : X \to \mathcal{G}(Y) \subseteq \Sigma_Y \to [0,1]$. We see that f is the curried version of a Markov kernel $f : X \times \Sigma_Y \to [0,1]$. Moreover, the condition that $f(x, \cdot)$ is a probability measure is satisfied because $f(x) \in \mathcal{G}(Y)$ (the set probability measures on Σ_Y) and the condition that $f(\cdot, B)$ is Σ_X-measurable is satisfied because

$$f(\cdot, B)^{-1}(M) = \{x \in X \mid f(x)(B) \in M\} = f^{-1}(N), \qquad N \subseteq \{p \in \mathcal{G}(Y) \mid p(B) \in M\},$$

and since N and f are measurable, so is f.

It remains to show that composition of Markov kernels corresponds to the Kleisli composition. Let $f : X \to \mathcal{G}(Y)$ and $g : Y \to \mathcal{G}(Y)$ are Kleisli morphisms, then recall that we have

$$g \circ_K f = \mu_Z \circ \mathcal{G}(g) \circ f = x \mapsto (C \mapsto \int_{z \in \mathcal{G}(Z)} e_C(z) f(x)(g^{-1}(dz))$$

With some rearrangements, namely uncurrying f and the composition as well as writing $e_C(z)$ as $z(C)$, we obtain:

$$g \circ_K f = (x, C) \mapsto \int_{z \in \mathcal{G}(Z)} z(C) f(x, g^{-1}(dz)).$$

This is clearly the formula for Markov kernel composition modulo the change of variable $z = g(y)$. $\square$

Remark 95. Recall that the Kleisli category of the power set monad $\mathcal{P} : \mathbf{Sets} \to \mathbf{Sets}$ is the category **Rel** of sets with relations as morphisms. The Giry monad is, in some sense, imitating the behavior of the power set for measurable spaces, thus we can think of Markov kernels as measurable relations or probabilistic relations.